

## Thermodynamic Products in Extended Phase Space

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We have examined the thermodynamic properties for a variety of spherically symmetric charged-AdS black hole (BH) solutions, including the charged AdS BH surrounded by quintessence dark energy and charged AdS BH in  $f(R)$  gravity in *extended phase-space*. This framework involves treating the cosmological constant as thermodynamic variable (for example: thermodynamic pressure and thermodynamic volume). Then they should behave as an analog of Van der Waal (VdW) like systems. In the extended phase space we have calculated the *entropy product* and *thermodynamic volume product* of all horizons. The mass (or enthalpy) independent nature of the said product signals they are *universal* quantities. The divergence of the specific heat indicates that the second order phase transition occurs under certain condition. In the appendix-A, we have studied the thermodynamic volume products for axisymmetric spacetime and it is shown to be *not universal* in nature. Finally, in appendix-B, we have studied the  $P - V$  criticality of Cauchy horizon for charged-AdS BH and found to be an universal relation of critical values between two horizons as  $P_c^- = P_c^+$ ,  $v_c^- = v_c^+$ ,  $T_c^- = -T_c^+$ ,  $\rho_c^- = -\rho_c^+$ . The symbols are defined in the main work.

*Keywords:* Entropy product, Thermodynamic volume product.

### 1. Introduction

An interesting topic in recent years both in the general relativity community and in the String theory community is that the BH area (or entropy) product formula of all horizons independent of the ADM (Arnowitt-Deser-Misner) mass of the background space-time<sup>1,2,3,4,5,6,7,8,9,10</sup>. For example, the area product formula for a Kerr BH<sup>1</sup> depends only on the angular momentum parameter:

$$\mathcal{A}_2 \mathcal{A}_1 = 64\pi^2 J^2 . \quad (1)$$

where  $\mathcal{A}_2$  and  $\mathcal{A}_1$  are area of the inner and outer horizons.

Whenever, we have taken the perturbed space-time with a spinning BH in some non-trivial environment e.g. a BH surrounded by a ring of matter or a multiple BH space-time the same formula holds. Hence, the area product formula of outer

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horizon or event horizon ( $\mathcal{H}^+$ ) and inner horizon or Cauchy horizons ( $\mathcal{H}^-$ ) for Kerr BH is of an *universal* quantity: it holds independently of the environment of the BH.

On the other hand, if we incorporate the BPS (Bogomol'ni-Prasad-Sommerfeld) states, the area product formula of  $\mathcal{H}^\mp$  should read <sup>2</sup>

$$\mathcal{A}_2 \mathcal{A}_1 = 64\pi^2 \ell_{pl}^4 N, \quad N \in \mathbb{N}, N_1 \in \mathbb{N}, N_2 \in \mathbb{N}. \quad (2)$$

where  $\ell_{pl}$  is the Planck length. This indicates the area product should be quantized.

Alternatively, the area products independent of mass implies that there should be an important role of the Cauchy horizon in the BH thermodynamics as well as in BH physics. Now the relevant question is that the mass independent product formula is generic? It has been shown explicitly by Visser <sup>6</sup> that by incorporating the cosmological constant, the area product of all physical horizons is not mass independent. But typically, some complicated function of inner and outer horizon area is indeed mass independent.

Previous studies have not made use of the extended phase space formalism. Thus in this work, we wish to examine the thermodynamic product formula in *extended phase space*. Where the ADM mass of an AdS BH could be treated as the enthalpy of the space-time and the cosmological constant should be treated as the thermodynamic pressure <sup>23</sup>. Therefore there must exist a conjugate quantity which is a thermodynamic volume associated with the BH space-time.

Besides area (or entropy) products, it needs to be evaluated whether other thermodynamic products <sup>8,9,11,12,13,14</sup> like BH temperature products, specific heat products, Komar energy products etc. are provide any universal characterization or not, and here we first introduced the *thermodynamic volume products* when one must consider the extended phase space thermodynamics. Does it independent of the ADM mass parameter? We will investigate this issues in the present work. So, when the cosmological constant treated as a thermodynamic pressure and its conjugate variable as a thermodynamic volume what happens the *Smarr mass formula*, *Smarr-Gibbs-Duhem* relation and BH *equations of state* in the extended phase space. Additionally, we find the mass independent volume products relation in parallel with the entropy product relations.

BH thermodynamic properties have been investigated for many decades and still it is going on. In the present study, the main motivation comes from the seminal work of Hawking and Page <sup>16</sup> where the thermodynamic properties of BHs in Schwarzschild-AdS space has been explicitly studied. The author discussed the phase transition (between small and large BHs for Schwarzschild-AdS BH) which is called famous Hawking-Page phase transition. The special interest is due to the application of AdS space-time in gauge/gravity duality via dual conformal field theory (CFT) through AdS/CFT correspondence <sup>17</sup>. Several exotic phenomena occurs in the AdS space-time. First example of course be Hawking-Page phase transition in Schwarzschild-AdS spacetime. The second one is that in charged AdS spacetime, the gravitational analogue of the liquid/gas phase transition has been observed in

the phase diagram which was explicitly investigated by several authors<sup>18,19,20,21</sup> and the fact that for charged AdS BH the notion of thermodynamic equilibrium is a straightforward concept. The third one is that Kerr-AdS spacetime admits reentrant phase transition and showing a tri-critical point in their phase diagram<sup>22</sup>.

The current interest is involved due to the variation of negative cosmological constant and also it is proportional to the thermodynamic pressure<sup>24,25,21</sup>. The thermodynamic products especially area (or entropy) products in charged AdS BHs were calculated in<sup>6</sup> but the author has not been considered there the extended phase space. Here we shall compute the volume products by considering the thermodynamic pressure ( $P$ ) is equal to the negative cosmological constant ( $\Lambda$ ) divided by  $8\pi$  (where  $G = c = \hbar = 1$ ) i.e.

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}. \quad (3)$$

and the corresponding thermodynamic volume can be defined as

$$V = \left( \frac{\partial M}{\partial P} \right)_{S,Q,J}. \quad (4)$$

This volume for charged-AdS BH should read as

$$V_i = \frac{4}{3}\pi r_i^3. \quad (5)$$

where  $r_i$  is the corresponding horizon radius and  $i = 1, 2, 3, 4$ .

It has been shown that the *Reverse Isoperimetric Inequality* is satisfied for event horizon<sup>26</sup>. Here we conjecture that this inequality is valid for all the horizons i.e.

$$\mathcal{R}_i = \left( \frac{3V_i}{4\pi} \right)^{\frac{1}{3}} \left( \frac{4\pi}{\mathcal{A}_i} \right)^{\frac{1}{2}} \geq 1. \quad (6)$$

It should be noted that a class of BHs with non-compact event horizons do not satisfy this inequality<sup>27,28</sup>.

The structure of the paper is as follows. In Sec.(2), we have described the thermodynamic properties of RN-AdS BH. The Sec.(3) describes the thermodynamic properties of the RN-AdS BH surrounded by quintessence. In Sec. (4), we have given the thermodynamic properties of the  $f(R)$  gravity. Finally, we conclude in Sec.(5). In appendix-A, we have examined the thermodynamic volume products for axisymmetric space-time and in appendix-B, we have investigated the  $P - V$  criticality of inner horizon for RN-AdS BH.

## 2. Thermodynamic properties of Charged AdS BH:

Let us begin with the charged-AdS space-time metric can be written as in terms of Schwarzschild like coordinates<sup>18,19</sup>:

$$ds^2 = -\mathcal{U}(r)dt^2 + \frac{dr^2}{\mathcal{U}(r)} + r^2 d\Omega_2^2. \quad (7)$$

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where,

$$\mathcal{U}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{\ell^2}, \quad (8)$$

and  $d\Omega_2^2$  is the metric on the unit sphere in two dimensions.

The electromagnetic potential one form for the space-time (7) is

$$A = A_\mu dx^\mu = -\frac{Q}{r} dt. \quad (9)$$

The BH horizons determined by the condition  $\mathcal{U}(r) = 0$  i.e.

$$\frac{r^4}{\ell^2} + r^2 - 2Mr + Q^2 = 0. \quad (10)$$

In terms of thermodynamic pressure, this could be rewritten as

$$\frac{8\pi P}{3} r^4 + r^2 - 2Mr + Q^2 = 0. \quad (11)$$

To finding the roots we apply the Vieta's rule, we get

$$\sum_{i=1}^4 r_i = 0. \quad (12)$$

$$\sum_{1 \leq i < j \leq 4} r_i r_j = \frac{3}{8\pi P}. \quad (13)$$

$$\sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = \frac{3M}{4\pi P}. \quad (14)$$

$$\prod_{i=1}^4 r_i = \frac{3Q^2}{8\pi P}. \quad (15)$$

The entropy of the BH can be defined as

$$\mathcal{S}_i = \frac{\mathcal{A}_i}{4}. \quad (16)$$

where the area of the BH is

$$\mathcal{A}_i = 4\pi r_i^2. \quad (17)$$

and now the BH temperature reads as

$$T_i = \frac{\mathcal{U}'(r)}{4\pi} = \frac{1}{4\pi r_i} \left( 1 + 8\pi P r_i^2 - \frac{Q^2}{r_i^2} \right). \quad (18)$$

The electric potential could be defined as

$$\Phi_i = \frac{Q}{r_i}. \quad (19)$$

We should be noted that in the extended phase space the ADM mass can be treated as the total gravitational enthalpy of the system i.e.  $M = H = U + PV$ .

Where  $U$  is the thermal energy of the system<sup>23</sup>. Then the first law of BH thermodynamics in the extended phase space becomes

$$dH = T_i d\mathcal{S}_i + V_i dP + \Phi_i dQ . \quad (20)$$

and the corresponding Smarr-Gibbs-Duhem relation becomes

$$H = 2T_i \mathcal{S}_i - 2PV_i + Q\Phi_i . \quad (21)$$

Now we compute the mass(or enthalpy) independent entropy sum and entropy product formula using Eqs.(12,13,15,16,17):

$$\sum_{i=1}^4 \sqrt{\mathcal{S}_i} = 0 . \quad (22)$$

$$\sum_{1 \leq i < j \leq 4} \sqrt{\mathcal{S}_i \mathcal{S}_j} = \frac{3}{8P} . \quad (23)$$

$$\prod_{i=1}^4 \sqrt{\mathcal{S}_i} = \frac{3\pi Q^2}{8P} . \quad (24)$$

In terms of two horizons, the mass-independent entropy product formula should read

$$\frac{\left(\frac{3Q^2}{8P}\right)}{\sqrt{\mathcal{S}_1 \mathcal{S}_2}} - \frac{\mathcal{S}_1 + \mathcal{S}_2 + \sqrt{\mathcal{S}_1 \mathcal{S}_2}}{\pi} = \frac{3}{8\pi P} . \quad (25)$$

Although it is a complicated function of two horizons but it is explicitly mass independent function of inner horizon and outer horizons.

Now we turn into another important relations that is the *volume sum* and *volume product* which are mass independent:

$$\sum_{i=1}^4 V_i^{\frac{1}{3}} = 0 . \quad (26)$$

$$\sum_{1 \leq i < j \leq 4} (V_i V_j)^{\frac{1}{3}} = \left(\frac{3}{32\pi}\right)^{\frac{1}{3}} \frac{1}{P} . \quad (27)$$

$$\prod_{i=1}^4 (V_i)^{\frac{1}{3}} = \left(\frac{\pi}{6}\right)^{\frac{1}{3}} \frac{Q^2}{P} . \quad (28)$$

Again in terms of two horizons, the mass independent volume product formula should be

$$\left(\frac{3}{32\pi}\right)^{\frac{1}{3}} \frac{\left(\frac{Q^2}{P}\right)}{(V_1 V_2)^{\frac{1}{3}}} - \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \left[V_1^{\frac{2}{3}} + V_2^{\frac{2}{3}} + (V_1 V_2)^{\frac{1}{3}}\right] = \frac{3}{8\pi P} . \quad (29)$$

These are explicitly mass-independent relations in the extended phase space.

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Finally, the equation of state in the extended phase space should read

$$P = \frac{T_i}{2r_i} - \frac{1}{8\pi r_i^2} + \frac{Q^2}{8\pi r_i^4} . \quad (30)$$

where  $r_i = \left(\frac{3V_i}{4\pi}\right)^{1/3}$ . Now in terms of specific volume  $v_i = 2r_i$  the above equation could be re-written as

$$P = \frac{T_i}{v_i} - \frac{1}{2\pi v_i^2} + \frac{2Q^2}{\pi v_i^4} . \quad (31)$$

The critical point can be obtained from the following conditions:

$$\frac{\partial P}{\partial v_i} = \frac{\partial^2 P}{\partial v_i^2} = 0 . \quad (32)$$

The critical values are explicitly computed in <sup>21</sup>. Defining further  $p = \frac{P}{P_c}$ ,  $\nu_i = \frac{v_i}{v_c}$  and  $\tau_i = \frac{T_i}{T_c}$ , the law of corresponding states become

$$8\tau_i = 3\nu_i \left( p + \frac{2}{\nu_i^2} \right) - \frac{1}{\nu_i^3} . \quad (33)$$

<sup>a</sup>. In the extended phase space the Gibbs free energy could be defined as

$$G_i = H - T_i S_i = M - T_i S_i = \frac{r_i}{4} - \frac{2\pi P}{3} r_i^3 + \frac{3Q^2}{4r_i} . \quad (34)$$

and the Helmholtz free energy is given by

$$F_i = G_i - P V_i = \frac{r_i}{2} - \pi T_i r_i^2 + \frac{Q^2}{2r_i} . \quad (35)$$

which is very important to determine the behavior of the critical exponents. <sup>b</sup> One may compute the entropy via the relation:

$$\mathcal{S}_i = - \left( \frac{\partial F_i}{\partial T_i} \right)_{V_i} = \pi r_i^2 . \quad (36)$$

which is exactly same as in Eq. (16). There are two types of specific heat. The specific heat at constant thermodynamic volume and the specific heat at constant pressure. They are defined as

$$(C_V)_i = T_i \left( \frac{\partial \mathcal{S}_i}{\partial T_i} \right)_V . \quad (37)$$

and

$$(C_P)_i = T_i \left( \frac{\partial \mathcal{S}_i}{\partial T_i} \right)_P . \quad (38)$$

<sup>a</sup>The critical values for charged-AdS BH determined in <sup>21</sup> are  $P_c = \frac{1}{96\pi Q^2}$ ,  $v_c = 2\sqrt{6}Q$  and  $T_c = \frac{\sqrt{6}}{18\pi Q}$

<sup>b</sup>The critical exponents for the BH system are  $\alpha, \beta, \gamma$  and  $\delta$ . The numerical values for charged-AdS BH are calculated in <sup>21</sup> as  $\alpha = 0$ ,  $\beta = 1/2$ ,  $\gamma = 1$  and  $\delta = 3$ .

From Eq. 36, we can easily see that the entropy  $\mathcal{S}_i$  is independent of  $T_i$  therefore we get

$$(C_V)_i = 0. \quad (39)$$

and we find

$$(C_P)_i = -2\pi r_i^2 \frac{\left(1 - \frac{Q^2}{r_i^2} + 8\pi P r_i^2\right)}{\left(1 - \frac{3Q^2}{r_i^2} - 8\pi P r_i^2\right)}. \quad (40)$$

The specific heat  $C_P$  diverges at

$$8\pi P r_i^4 - r_i^2 + 3Q^2 = 0. \quad (41)$$

or i.e. at

$$r_i = \pm \sqrt{\frac{1 \pm \sqrt{1 - 96\pi Q^2 P}}{16\pi P}}. \quad (42)$$

which implies a second order phase transition occurs at that point.

### 3. Thermodynamic properties of the RN-AdS BH surrounded by quintessence:

In this section, we will show how the quintessence dark energy matter does affect on the thermodynamic product relation in the extended phase space. The metric function of Eq. (7) for RN-AdS BH surrounded by quintessence can be written as

$$\mathcal{U}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{a}{r^{3w_q+1}} - \frac{\Lambda}{3} r^2. \quad (43)$$

where  $w_q$  is the state parameter and  $a$  is the normalization factor related to the density of quintessence. The ranges for quintessence dark energy is  $-1 < w_q < -\frac{1}{3}$  and for phantom dark energy:  $w_q < -1$ . In terms of  $a$ , the density of quintessence can be defined as

$$\rho_q = -\frac{3aw_q}{2r^{3w_q+1}}. \quad (44)$$

In the extended phase space the function can be written as

$$\mathcal{U}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{a}{r^{3w_q+1}} + \frac{8\pi P}{3} r^2. \quad (45)$$

Now the horizon Eq. can be written as

$$\frac{8\pi P}{3} r^{3w_q+3} + r^{3w_q+1} - 2Mr^{3w_q} + Q^2 r^{3w_q-1} - a = 0. \quad (46)$$

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Using Vieta's theorem, we find

$$\sum_{i=1}^{3w_q+3} r_i = 0 . \quad (47)$$

$$\sum_{1 \leq i < j \leq (3w_q+3)} r_i r_j = \frac{3}{8\pi P} . \quad (48)$$

$$\sum_{1 \leq i < j < k \leq (3w_q+3)} r_i r_j r_k = \frac{3M}{4\pi P} . \quad (49)$$

$$\sum_{1 \leq i < j < k < l \leq (3w_q+3)} r_i r_j r_k r_l = \frac{3Q^2}{8\pi P} . \quad (50)$$

$$\prod_{i=1}^{3w_q+3} r_i = a . \quad (51)$$

It should be mentioned that  $3w_q$  is an integer quantity.

The entropy  $\mathcal{S}_i$  and electric potential  $\Phi_i$  are same as in RN-AdS case. Now the mass of the BH could be expressed in terms of the horizon radius and dynamic pressure:

$$M = \frac{r_i}{2} \left( 1 + \frac{Q^2}{r_i^2} - \frac{a}{r_i^{3w_q+1}} + \frac{8\pi P}{3} r_i^2 \right) . \quad (52)$$

Hence the first law of thermodynamics becomes

$$dH = T_i d\mathcal{S}_i + V_i dP + \Phi_i dQ + \mathcal{A}_i da . \quad (53)$$

where  $\mathcal{A}_i = \left( \frac{\partial H}{\partial a} \right)_{S_i, Q, P} = -\frac{1}{2r_i^{3w_q}}$  is defined to be a physical quantity conjugate to the state parameter <sup>31</sup>. The corresponding Smarr relation reads

$$H = 2T_i \mathcal{S}_i - 2PV_i + Q\Phi_i + (1 + 3w_q)\mathcal{A}_i da . \quad (54)$$

Again the mass(or enthalpy) independent entropy sum and entropy product relations are

$$\sum_{i=1}^{(3w_q+3)} \sqrt{\mathcal{S}_i} = 0 . \quad (55)$$

$$\sum_{1 \leq i < j \leq (3w_q+3)} \sqrt{\mathcal{S}_i \mathcal{S}_j} = \frac{3}{8P} . \quad (56)$$

$$\prod_{i=1}^{(3w_q+3)} \sqrt{\frac{\mathcal{S}_i}{\pi}} = a . \quad (57)$$



Similarly, the mass independent volume sum and volume product relations are

$$\sum_{i=1}^{(3w_q+3)} V_i^{\frac{1}{3}} = 0. \quad (58)$$

$$\sum_{1 \leq i < j \leq (3w_q+3)} (V_i V_j)^{\frac{1}{3}} = (4\pi)^{\frac{2}{3}} \frac{3}{8\pi P}. \quad (59)$$

$$\prod_{i=1}^{(3w_q+3)} \left( \frac{3V_i}{4\pi} \right)^{\frac{1}{3}} = a. \quad (60)$$

These are explicitly mass-independent relations for RN-AdS BH surrounded by quintessence. It follows from the above analysis that the entropy product and volume product relations are strictly dependent on *quintessence dark energy matter*. It is quite interesting to mention that the entropy product is mass-independent but there has been effect of quintessence dark energy matter on that thermodynamic product relations.

Now the BH temperature for all the horizons could be defined as

$$T_i = \frac{1}{4\pi r_i} \left( 1 + 8\pi P r_i^2 - \frac{Q^2}{r_i^2} + \frac{3aw_q}{r_i^{3w_q+1}} \right). \quad (61)$$

and the BH equation of state should read

$$P = \frac{T_i}{2r_i} - \frac{1}{8\pi r_i^2} + \frac{Q^2}{8\pi r_i^4} - \frac{3aw_q}{8\pi r_i^{3(w_q+1)}}. \quad (62)$$

where  $r_i = \left( \frac{3V_i}{4\pi} \right)^{1/3}$ . Again in terms of specific volume  $v_i = 2r_i$  the above equation should be rewritten as

$$P = \frac{T_i}{v_i} - \frac{1}{2\pi v_i^2} + \frac{2Q^2}{\pi v_i^4} - \frac{3aw_q 2^{4w_q}}{\pi v_i^{3(w_q+1)}}. \quad (63)$$

The critical values are explicitly calculated in <sup>31</sup>. So we do not write here. The Gibbs free energy for all the horizons could be written as

$$G_i = H - T_i S_i = M - T_i S_i = \frac{r_i}{4} - \frac{2\pi P}{3} r_i^3 + \frac{3Q^2}{4r_i} - \frac{3aw_q + 2a}{4r_i^{3w_q}}. \quad (64)$$

Again we compute the specific heat at constant thermodynamic pressure to study the local stability of the BH given by

$$(C_P)_i = -2\pi r_i^2 \frac{\left( 1 - \frac{Q^2}{r_i^2} + \frac{3aw_q}{r_i^{3w_q+1}} + 8\pi P r_i^2 \right)}{\left( 1 - \frac{3Q^2}{r_i^2} + \frac{3(2+3w_q)aw_q}{r_i^{3w_q+1}} - 8\pi P r_i^2 \right)}. \quad (65)$$

It should be noted that the specific heat diverges at

$$1 - \frac{3Q^2}{r_i^2} + \frac{3(2+3w_q)aw_q}{r_i^{3w_q+1}} - 8\pi P r_i^2 = 0. \quad (66)$$

which signals a second order phase transition.

#### 4. Thermodynamic properties of AdS BH in $f(R)$ gravity:

This section is dedicated to study the thermodynamic properties of a static, spherically symmetric AdS BH in  $f(R)$  gravity. It is a kind of modified gravity and it is very useful tool for explaining the current and future state of the accelerating universe. It is also helpful for explaining the inflation and structure formation in the early universe. The metric <sup>29,30</sup> function for this kind of gravity can be written as

$$\mathcal{U}(r) = 1 - \frac{2m}{r} + \frac{q^2}{\alpha r^2} - \frac{R_0}{12} r^2. \quad (67)$$

where  $\alpha = 1 + f'(R_0)$ . The quantities  $m$  and  $q$  are related to the  $M$  (ADM mass) and  $Q$  (electric charge) in this gravity becomes

$$M = m\alpha, \quad Q = \frac{q}{\sqrt{\alpha}}. \quad (68)$$

As is the thermodynamic pressure in  $f(R)$  gravity can be written as  $P = -\frac{\Lambda}{8\pi}\alpha$  and the constant scalar curvature as  $R_0 = -\frac{12}{\ell^2} = 4\Lambda$ . The corresponding thermodynamic volume can be defined as  $V_i = \frac{4}{3}\pi r_i^3$ . Therefore the horizon equation should read

$$\frac{8\pi P}{3}\alpha r^4 + \alpha r^2 - 2m\alpha r + q^2 = 0. \quad (69)$$

To finding the roots we again apply the Vieta's rule, we have

$$\sum_{i=1}^4 r_i = 0. \quad (70)$$

$$\sum_{1 \leq i < j \leq 4} r_i r_j = \frac{3}{8\pi P}. \quad (71)$$

$$\sum_{1 \leq i < j < k \leq 4} r_i r_j r_k = \frac{3m}{4\pi P}. \quad (72)$$

$$\prod_{i=1}^4 r_i = \frac{3q^2}{8\pi\alpha P}. \quad (73)$$

The entropy can be defined as

$$\mathcal{S}_i = \pi\alpha r_i^2. \quad (74)$$

and the BH temperature<sup>30</sup> should be

$$T_i = \frac{1}{4\pi r_i} \left( 1 + \frac{8\pi P}{\alpha} r_i^2 - \frac{q^2}{\alpha r_i^2} \right). \quad (75)$$

Again the electric potential in  $f(R)$  gravity could be defined as

$$\Phi_i = \frac{q}{r_i} \sqrt{\alpha}. \quad (76)$$

Now the mass(or enthalpy) independent entropy sum and entropy product formula in  $f(R)$  gravity should read:

$$\sum_{i=1}^4 \sqrt{\mathcal{S}_i} = 0 . \quad (77)$$

$$\sum_{1 \leq i < j \leq 4} \sqrt{\mathcal{S}_i \mathcal{S}_j} = \frac{3\alpha}{8P} . \quad (78)$$

$$\prod_{i=1}^4 \sqrt{\mathcal{S}_i} = \frac{3\pi\alpha q^2}{8P} . \quad (79)$$

In terms of two horizons, the mass-independent entropy product formula reads as

$$\frac{\left(\frac{3q^2}{8P}\right)}{\sqrt{\mathcal{S}_1 \mathcal{S}_2}} - \frac{\mathcal{S}_1 + \mathcal{S}_2 + \sqrt{\mathcal{S}_1 \mathcal{S}_2}}{\pi\alpha} = \frac{3}{8\pi P} . \quad (80)$$

Again the mass independent *volume sum* and *volume product* becomes

$$\sum_{i=1}^4 V_i^{\frac{1}{3}} = 0 . \quad (81)$$

$$\sum_{1 \leq i < j \leq 4} (V_i V_j)^{\frac{1}{3}} = \left(\frac{3}{32\pi}\right)^{\frac{1}{3}} \frac{1}{P} . \quad (82)$$

$$\prod_{i=1}^4 (V_i)^{\frac{1}{3}} = \left(\frac{\pi}{6}\right)^{\frac{1}{3}} \frac{q^2}{\alpha P} . \quad (83)$$

Again in terms of two horizons, the mass independent volume product formula reads

$$\left(\frac{3}{32\pi}\right)^{\frac{1}{3}} \frac{\left(\frac{q^2}{P}\right)}{\alpha(V_1 V_2)^{\frac{1}{3}}} - \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \left[V_1^{\frac{2}{3}} + V_2^{\frac{2}{3}} + (V_1 V_2)^{\frac{1}{3}}\right] = \frac{3}{8\pi P} . \quad (84)$$

Once again these are explicitly mass-independent relation in the extended phase space in  $f(R)$  gravity. It should be noted that in the limit  $\alpha = 1$ , one obtains the result of RN-AdS BH in extended phase space. For our record we should be noted that the equation of state in  $f(R)$  gravity<sup>30</sup>:

$$P = \frac{\alpha T_i}{2r_i} - \frac{\alpha}{8\pi r_i^2} + \frac{q^2}{8\pi r_i^4} . \quad (85)$$

In terms of specific volume  $v_i = 2r_i$ , the above Eq. could be rewritten as

$$P = \frac{\alpha T_i}{v_i} - \frac{\alpha}{2\pi v_i^2} + \frac{2q^2}{\pi v_i^4} . \quad (86)$$

From the equation of state we can easily derived the critical constants by applying the appropriate condition. <sup>c</sup>.

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<sup>c</sup>The critical values for  $f(R)$  gravity explicitly computed in <sup>30</sup> are  $P_c = \frac{\alpha^2}{96\pi q^2}$ ,  $v_c = \frac{2q\sqrt{6}}{\sqrt{\alpha}}$  and  $T_c = \frac{\sqrt{6\alpha}}{18\pi q}$

Finally, the Gibbs free energy<sup>30</sup> should read

$$G_i = \frac{\alpha r_i}{4} - \frac{2\pi P}{3} r_i^3 + \frac{3q^2}{4r_i} . \quad (87)$$

In this case, the specific heat at constant pressure is found to be

$$(C_P)_i = -2\pi r_i^2 \frac{\left(1 - \frac{q^2}{\alpha r_i^2} + 8\pi P r_i^2\right)}{\left(1 - \frac{3q^2}{\alpha r_i^2} - 8\pi P r_i^2\right)} . \quad (88)$$

The specific heat diverges at

$$8\pi\alpha P r_i^4 - \alpha r_i^2 + 3q^2 = 0 . \quad (89)$$

or i.e. at

$$r_i = \pm \sqrt{\frac{\alpha \pm \sqrt{\alpha^2 - 96\pi\alpha P q^2}}{16\pi\alpha P}} . \quad (90)$$

Again it signals a second order phase transition.

## 5. Conclusion:

The present study demonstrated that the thermodynamic properties of spherically symmetric charged-AdS black hole, charged AdS BH surrounded by quintessence and charged AdS BH in  $f(R)$  gravity in the extended phase-space. The extended phase space means where the cosmological constant should be treated as thermodynamic pressure and its conjugate variable as a thermodynamic volume. We derived various thermodynamic products particularly entropy products and thermodynamic volume products. In all the three cases, it has been shown that the mass(or enthalpy) independent properties turn out to be an universal like quantities. It should be noted that the presence of the quintessence matter does affect on the expression of entropy product and thermodynamic volume products. The first law of BH thermodynamics and Smarr formula have been studied for all the horizons. The BH equation of state has been derived for all the horizons. The divergence of the specific heat indicates that the second order phase transition should occur at a certain condition. In summary, the thermodynamic relations that we derived provide some universal characterization of the BH which *may* provide insight into the origin of BH entropy both *inner and outer*.

## Appendix A.

In the main work, we have considered the thermodynamic volume for different spherically symmetric cases where it is related to the entropy via a proportionality of the type  $V_i \propto r_i \mathcal{S}_i$  and we proved that for each cases, the thermodynamic product relations are *independent of mass* thus the relations are *universal* in nature. Now

here we shall show what happens in case of axisymmetric cases where the thermodynamic volume is *not* directly proportional to the above mentioned relation? For example, we have taken the Kerr BH. The thermodynamic quantities for all the horizons  $\mathcal{H}^\pm$  are

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \quad \mathcal{A}_\pm = 4\pi(r_\pm^2 + a^2), \quad (\text{A.1})$$

$$S_\pm = \pi(r_\pm^2 + a^2), \quad T_\pm = \frac{r_\pm - r_\mp}{4\pi(r_\pm^2 + a^2)}, \quad (\text{A.2})$$

$$\Omega_\pm = \frac{a}{2Mr_\pm}. \quad (\text{A.3})$$

and the thermodynamic volume<sup>32</sup> for  $\mathcal{H}^+$  is

$$V_+ = \frac{\mathcal{A}_+ r_+}{3} \left[ 1 + \frac{a^2}{2r_+^2} \right]. \quad (\text{A.4})$$

and we claim that the thermodynamic volume for  $\mathcal{H}^-$  is defined to be

$$V_- = \frac{\mathcal{A}_- r_-}{3} \left[ 1 + \frac{a^2}{2r_-^2} \right]. \quad (\text{A.5})$$

We find an interesting relation between volume products and area products for an axisymmetric spacetime having two physical horizons namely  $\mathcal{H}^+$  and  $\mathcal{H}^-$ :

$$V_+ V_- = \frac{\mathcal{A}_+ \mathcal{A}_-}{18} (r_+ + r_-)^2. \quad (\text{A.6})$$

The main potential point of interest here is that the product of thermodynamic volume of  $\mathcal{H}^\pm$  and it is found to be for Kerr BH:

$$V_+ V_- = \frac{128}{9} \pi^2 J^2 M^2. \quad (\text{A.7})$$

It indicates that the thermodynamic volume product does *depend* on the mass parameter and the universality that we have found in spherically symmetric cases breaks down for axisymmetric cases. Thus the conclusion is that although the area (or entropy product) of  $\mathcal{H}^\pm$  is universal for simple Kerr BH but the volume product is *not universal*. The result should be valid for KN BH as well and the volume product is found to be

$$V_+ V_- = \frac{128}{9} \pi^2 \left( J^2 + \frac{Q^4}{4} \right) M^2. \quad (\text{A.8})$$

Therefore, it can be easily extend to Kerr-AdS BH and KN-AdS BH also.

## Appendix B.

In this section, we shortly introduce  $P - V$  criticality of Cauchy horizon for charged AdS BH. Does the inner horizon obey an equation of state? What happens to the inner horizons during phase transition has been discussed in<sup>33</sup>. Here we are interested to show what happens the BH equation of state in case of Cauchy horizon? What are the values of critical constant for this horizon does it same as is for event

horizon. This is the main interest here. Since we have taken charged AdS space-time and we have assumed that the BH should have at least two physical horizons. The outer horizon ( $r_+$ ) and inner horizon ( $r_-$ ). Then we find the relevant thermodynamic quantities for  $\mathcal{H}^+$  <sup>21</sup>:

$$\begin{aligned} \mathcal{A}_+ &= 4\pi r_+^2, \mathcal{S}_+ = \pi r_+^2, \Phi_+ = \frac{Q}{r_+}, V_+ = \frac{4}{3}\pi r_+^3, T_+ = \frac{1}{4\pi r_+} \left( 1 + 8\pi P r_+^2 - \frac{Q^2}{r_+^2} \right) \\ G_+ &= M - T_+ S_+ = M - \pi r_+^2 T_+, F_+ = G_+ - P V_+, dH = T_+ d\mathcal{S}_+ + V_+ dP + \Phi_+ dQ, \\ P &= \frac{T_+}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4}, r_+ = \left( \frac{3V_+}{4\pi} \right)^{1/3} \\ P &= \frac{T_+}{v_+} - \frac{1}{2\pi v_+^2} + \frac{2Q^2}{\pi v_+^4}, v_+ = 2r_+. \end{aligned} \quad (\text{B.1})$$

The critical point can be obtained at the point of inflection of  $\mathcal{H}^+$ :

$$\frac{\partial P}{\partial v_+} = \frac{\partial^2 P}{\partial v_+^2} = 0. \quad (\text{B.2})$$

and the critical values explicitly derived in <sup>21</sup> are

$$P_c^+ = \frac{1}{96\pi Q^2}, v_c^+ = 2\sqrt{6}Q, T_c^+ = \frac{\sqrt{6}}{18\pi Q}. \quad (\text{B.3})$$

Now we have derived the thermodynamic quantities for  $\mathcal{H}^-$ :

$$\begin{aligned} \mathcal{A}_- &= 4\pi r_-^2, \mathcal{S}_- = \pi r_-^2, \Phi_- = \frac{Q}{r_-}, V_- = \frac{4}{3}\pi r_-^3, T_- = -\frac{1}{4\pi r_-} \left( 1 + 8\pi P r_-^2 - \frac{Q^2}{r_-^2} \right) \\ G_- &= -M - T_- S_- = -M - \pi r_-^2 T_-, F_- = G_- - P V_-, -dH = -T_- d\mathcal{S}_- + V_- dP + \Phi_- dQ, \\ P &= -\frac{T_-}{2r_-} - \frac{1}{8\pi r_-^2} + \frac{Q^2}{8\pi r_-^4}, r_- = \left( \frac{3V_-}{4\pi} \right)^{1/3} \\ P &= -\frac{T_-}{v_-} - \frac{1}{2\pi v_-^2} + \frac{2Q^2}{\pi v_-^4}, v_- = 2r_-. \end{aligned} \quad (\text{B.4})$$

Similarly, we can compute the critical values for  $\mathcal{H}^-$  by applying the condition at the point of inflection:

$$\frac{\partial P}{\partial v_-} = \frac{\partial^2 P}{\partial v_-^2} = 0. \quad (\text{B.5})$$

and the critical values we find

$$P_c^- = \frac{1}{96\pi Q^2}, v_c^- = 2\sqrt{6}Q, \text{ and } T_c^- = -\frac{\sqrt{6}}{18\pi Q}. \quad (\text{B.6})$$

The only difference in critical values between two horizons is

$$T_c = -\frac{\sqrt{6}}{18\pi Q} . \quad (\text{B.7})$$

The critical temperature for  $\mathcal{H}^-$  is negative but other values are same. It is also true that the  $P - V$  diagram is qualitatively different for  $\mathcal{H}^+$  and  $\mathcal{H}^-$ . The critical ratio for  $\mathcal{H}^-$  is calculated to be

$$\rho_c^- = \frac{P_c v_c}{T_c} = -\frac{3}{8} . \quad (\text{B.8})$$

where as for  $\mathcal{H}^+$ ,  $\rho_c^+ = \frac{3}{8}$ . Thus, we conclude that

$$P_c^- = P_c^+ \quad (\text{B.9})$$

$$v_c^- = v_c^+ \quad (\text{B.10})$$

$$T_c^- = -T_c^+ . \quad (\text{B.11})$$

$$\rho_c^- = -\rho_c^+ . \quad (\text{B.12})$$

Finally, using the properties of symmetry in nature of  $r_{\pm}$ , one obtains the following thermodynamic quantities at  $\mathcal{H}^-$ :

$$\mathcal{A}_- = \mathcal{A}_+|_{r_+ \leftrightarrow r_-}, \mathcal{S}_- = \mathcal{S}_+|_{r_+ \leftrightarrow r_-}, \Omega_- = \Omega_+|_{r_+ \leftrightarrow r_-}, \Phi_- = \Phi_+|_{r_+ \leftrightarrow r_-} \quad (\text{B.13})$$

$$T_- = -T_+|_{r_+ \leftrightarrow r_-}, V_- = V_+|_{r_+ \leftrightarrow r_-}, G_- = -G_+|_{r_+ \leftrightarrow r_-} . \quad (\text{B.14})$$

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